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On some sequence spaces and A- statistical convergence

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Abstract

In the present paper, we introduce and study some properties of the a new sequence space that is defined using the φ- function and de la Valee-Poussin mean.

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Abstract. In the present paper, we introduce and study some properties of the a new sequence space that is defined using the φ- function and de la Valee-Poussin mean.

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I. Introduction and Background

Let w denote the set of all real and complex sequences $x = (x_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $||x|| = \sup_n |x_n|$, respectively. A sequence $x \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [4] has shown that

$$\hat{c} = \left\{ x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n \right\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + x_{n+2} + \cdots + x_{n+m}}{m+1}.$$

The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox [5] as follows :

$$[\hat{c}] = \left\{ x \in l_\infty : \lim_m t_{m,n}(|x - L|) = 0, \text{ uniformly in } n, \text{ for some } \ell \right\}.$$

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean if and only if

1. $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_n \geq 0$ for all n;
2. $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

Let V_σ denote the set of bounded sequences which have unique σ-mean (see, [16]).

It was quite natural to expect that invariant mean must give rise to a new type of convergence, namely , strong invariant convergence, just as almost convergence gives rise to the concept of strong almost convergence and this concept was introduced and discussed by

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Mursaleen [8]. If $[V_\sigma]$ denotes the set of all strongly σ-convergent sequences, then Mursaleen defined,

$$[V_\sigma] = \{x \in l_\infty : lim_m \frac{1}{m}\sum_{k=1}^{m} \left|x_{\sigma^k(n)} - l\right| = 0, \text{uniformly in } n\}.$$

Here $\sigma^k(n)$ denotes the k th iterate of the mapping σ at n. Taking $\sigma(n) = n+1$, we obtain $[V_\sigma] = [\hat{c}]$ so that strong σ- convergence generalizes the concept of strong almost convergence.

Let $\lambda = (\lambda_i)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0.$$

The generalized de la Valée-Poussin mean is defined by

$$T_i(x) = \frac{1}{\lambda_i}\sum_{k\in I_i} x_k$$

where $I_i = [i - \lambda_i + 1, i]$. A sequence $x = (x_n)$ is said to be (V,λ)-summable to a number L, if $T_i(x) \to L$ as $i \to \infty$ (see [6]) .

Quite Recently Malkowsky and Savaş [6] have introduced the space $[V,\lambda]$ of λ-strongly convergent sequences as follows:

$$[V,\lambda] = \left\{ x = (x_k) : \lim_i \frac{1}{\lambda_i}\sum_{k\in I_i} |x_k - L| = 0, \text{ for some } L \right\}.$$

Note that in the special case where $\lambda_i = i$, the space $[V,\lambda]$ reduces the space w of strongly Cesaro summable sequences which is defined by

$$w = \left\{ x = (x_k) : \lim_i \frac{1}{i}\sum_{k=1}^{i} |x_k - L|) = 0, \text{ for some } l \right\}.$$

Ruckle [10] used the idea of a modulus function f to construct a class of FK spaces

$$L(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f\left(|x_k|\right) < \infty \right\}$$

The space $L(f)$ is closely related to the space l_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.

Recently E. Savas [13] generalized the concept of strong almost convergence by using a modulus f and examined some properties of the corresponding new sequence spaces.

Following Ruckle, a modulus function f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$,
(iii) f increasing,
(iv) f is continuous from the right at zero.

Since $|f(x) - f(y)| \leq f\left(|x-y|\right)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. Furthermore, we have $f(nx) \leq nf(x)$ for all $n \in N$, from condition (ii) and so

$$\begin{aligned} f(x) &= f\left(nx\frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right) \quad \text{hence} \\ \frac{1}{n}f(x) &\leq f\left(\frac{x}{n}\right) \qquad \text{for all } n \in N \end{aligned}$$

A modulus may be bounded or unbounded. For example, $f(x) = x^p$, for $0 < p \leq 1$ is unbounded, but $f(x) = \dfrac{x}{1+x}$ is bounded.

By a φ-function we understand a continuous non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0, \varphi(u) > 0$,for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$, (see, [18]).
A φ-function φ is called non weaker than a φ-function ψ and we write $\psi \prec \varphi$ if there are constants $c, b, k, l > 0$ such that $c\psi(lu) \leq b\varphi(ku)$, (for all large or small u, respectively).
A φ-function φ and ψ are called equivalent and we write $\varphi \sim \psi$ if there are positive constant b_1, b_2, c, k_1, k_2, l such that $b_1\varphi(k_1u) \leq c\psi(lu) \leq b_2\varphi(k_2u)$, (for all large or small u, respectively), (see, [18]).
A φ-function φ is said to satisfy the condition (Δ_2),(for all large or small u, respectively) if for some constant $k > 1$ there is satisfied the inequality $\varphi(2u) \leq k\varphi(u)$.

In this paper, we introduce and study some properties of the following sequence space that is defined using the φ- function and de la Valée-Poussin mean and also some inclusion theorems are obtained.

II. Main Results

Let $\Lambda = (\lambda_n)$ be same in the above, φ be given φ-function and f be given modulus function, respectively. Moreover, let an infinite matrix $A = (a_{nk})$ be given. Then we write,

$$V_\lambda^0((A,\varphi,\sigma), f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n\in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

If $x \in V_\lambda^0((A,\varphi,\sigma), f)$, the sequence x is said to be λ -strong (A,φ,σ)- convergent to zero with respect to a modulus f. When $\varphi(x) = x$ for all x, we obtain,

$$V_\lambda^0((A,\sigma), f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n\in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}(|x_{\sigma^k(i)}|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

If $f(x) = x$, we write

$$V_\lambda^0(A,\varphi,\sigma) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n\in I_j} \Big(\Big| \sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

If we take $A = I$ and $\varphi(x) = x$ respectively, then we have

$$V_\lambda^0((I,\sigma), f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{k\in I_j} f\Big(\big|x_{\sigma^k(i)}\big| \Big) = 0, \text{uniformly in } i \right\}.$$

If we take $A = I$, $\varphi(x) = x$ and $f(x) = x$ respectively, then we have

$$V_\lambda^0((I,\sigma)) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{k \in I_j} |x_{\sigma^k(i)}| = 0, \text{ uniformly in } i \right\},$$

which was defined and studied by Savaş and Savaş [12]. If we define the matrix $A = (a_{nk})$ as follows:

$$a_{nk} = \frac{1}{n} \text{ for } n \geq k \text{ and } a_{nk} = 0 \text{ otherwise},$$

then we have the sequence space,

$$V_\lambda^0((C,\varphi,\sigma),f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} f\Big(\Big| \frac{1}{n} \sum_{k=1}^{n} \varphi(|x_{\sigma^k(i)}|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

If we take $\sigma(i) = i + 1$, the sequence spaces $V_\lambda^0((A,\varphi,\sigma),f)$,$V_\lambda^0((A,\sigma),f)$, $V_\lambda^0((A,\varphi,\sigma))$, $V_\lambda^0((I,\sigma),f)$ and $V_\lambda^0((C,\varphi,\sigma),f)$ reduce to the following sequence spaces respectively.

$$\hat{V}_\lambda^0((A,\varphi),f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}\varphi(|x_{k+i}|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

$$\hat{V}_\lambda^0(A,f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}(|x_{k+i}|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

$$\hat{V}_\lambda^0(A,\varphi) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} \Big(\Big| \sum_{k=1}^{\infty} a_{nk}\varphi(|x_{k+i}|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

$$\hat{V}_\lambda^0(I,f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{k \in I_j} f(|x_{k+i}|) = 0, \text{ uniformly in } i \right\}.$$

which was defined and studied by Malkowsky and Savas [6].

$$\hat{V}_\lambda^0((C,\varphi),f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{k \in I_j} f\Big(\Big| \frac{1}{n} \sum_{k=1}^{n} \varphi(|x_{k+i}|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

We now have,

Theorem II.1. *Let $A = (a_{nk})$ be an infinite matrix and let the φ - function $\varphi(u)$ satisfies the condition (Δ_2). Then the following conditions are true.*
(a) If $x = (x_k) \in V_\lambda^0((A,\varphi,\sigma),f)$ and α is an arbitrary number, then $\alpha x \in V_\lambda^0((A,\varphi,\sigma),f)$.

(b) If $x, y \in V_\lambda^0((A,\varphi,\sigma), f)$ where $x = (x_k)$, $y = (y_k)$ and α, β are given numbers, then $\alpha x + \beta y \in V_\lambda^0((A,\varphi,\sigma), f)$.

Proof. (a) Let $x \in V_\lambda^0((A,\varphi,\sigma), f)$. First let us remark that for $0 < \alpha < 1$ we write

$$\frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|\alpha x_{\sigma^k(i)}|)\Big|\Big) \le \frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|)\Big|\Big).$$

Furthermore, if $\alpha > 1$ then we may find a positive number s such that $\alpha < 2^s$ and we have

$$\begin{aligned}\frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|\alpha x_{\sigma^k(i)}|)\Big|\Big)&\\ &\le \frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(d^s\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|)\Big|\Big),\\ &\le \frac{L}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|)\Big|\Big),\end{aligned}$$

where d and L are constants connected with the properties of φ and f. Therefore we obtain the condition (a).
(b) In the following let the numbers α, β and the elements $x, y \in V_\lambda^0((A,\varphi,\sigma), f)$ be given. From the part (a) it follows that the following inequality is true

$$\begin{aligned}\frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|\alpha x_{\sigma^k(i)} + \beta x_{\sigma^k(i)}|)\Big|\Big)&\\ &\le L_1\frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|)\Big|\Big)\\ &+ L_2\frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|)\Big|\Big),\end{aligned}$$

where the constants L_1 and L_2 are defined as in (a). Hence $\alpha x + \beta y \in V_\lambda^0((A,\varphi,\sigma), f)$.

■

Theorem II.2. *Let $f$be an modulus function. Then*

$$V_\lambda^0(A,\varphi,\sigma) \subset V_\lambda^0((A,\varphi,\sigma), f).$$

Proof. Let $x \in V_\lambda^0(A,\varphi,\sigma)$. For a given $\varepsilon > 0$ we choose $0 < \delta < 1$ such that $f(x) < \varepsilon$ for every $x \in [0,\delta]$. We can write

$$\frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|)\Big|\Big) = S_1 + S_2$$

,
where $S_1 = \frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\big|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|)\big|\Big)$ and this sum is taken over

$$\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|) \le \delta$$

and

$$S_2 = \frac{1}{\lambda_j} \sum_{n \in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk} \varphi(|x_{\sigma^k(i)}|) \Big| \Big)$$

and this sum is taken over

$$\sum_{k=1}^{\infty} a_{nk} \varphi(|x_{\sigma^k(i)}|) > \delta.$$

By definition of the modulus f we have $S_1 = \frac{1}{\lambda_j} \sum_{n \in I_j} f\Big(\delta\Big) = f(\delta) < \varepsilon$ and furthermore

$$S_2 = f(1) \frac{1}{\delta} \frac{1}{\lambda_j} \sum_{n \in I_j} \sum_{k=1}^{\infty} a_{nk} \varphi(|x_{\sigma^k(i)}|).$$

Hence we have $x \in V_\lambda^0((A, \varphi, \sigma), f)$.
This completes the proof.

■

III. $A-$ statistical convergence

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [1] (see also Schoenberg [17]) as follows :

A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero, i.e.,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \chi_K(\varepsilon)(k) = 0,$$

where $\chi_K(\varepsilon)$ denotes the characteristic function of $K(\varepsilon)$.

Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [2] and Šalát [11]. Di Maio and Kočinac [7] introduced the concept of statistical convergence in topological spaces and statistical Cauchy condition in uniform spaces and established the topological nature of this convergence.

In another direction, a new type of convergence called λ- statistical convergence was introduced in [8] as follows.

A sequence (x_k) of real numbers is said to be λ- statistically convergent to L (or, S_λ-convergent to L) if for any $\epsilon > 0$,

$$\lim_{j \to \infty} \frac{1}{\lambda_j} |\{k \in I_j : |x_k - L| \geq \epsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [8] the relation between λ- statistical convergence and statistical convergence was established among other things.

Recently E. Savas [15] defined almost λ-statistical convergence by using the notion of (V, λ)-summability to generalize the concept of statistical convergence.

Let the infinite matrix $A = (a_{nk})$, the sequence $x = (x_k)$, the φ- function $\varphi(u)$ and a positive number $\varepsilon > 0$ be given. We write, for all i

$$K_\lambda^j((A, \varphi, \sigma), \varepsilon) = \{n \in I_j : \sum_{k=1}^{\infty} a_{nk} \varphi(|x_{\sigma^k(i)}|) \geq \varepsilon\}.$$

The sequence x is said to be (A,φ,σ)- statistically convergent to a number zero if for every $\varepsilon > 0$

$$lim_r \frac{1}{\lambda_j}\mu(K_\lambda^j((A,\varphi,\sigma),\varepsilon)) = 0, \text{uniformly in } i$$

where $\mu(K_\lambda^j((A,\varphi,\sigma),\varepsilon))$ denotes the number of elements belonging to $K_\lambda^j((A,\varphi,\sigma),\varepsilon)$. We denote by $S_\lambda^0((A,\varphi,\sigma))$, the set of sequences $x=(x_k)$ which are uniform $(A,\varphi,\sigma)-$ statistical convergent to zero. We write

$$S_\lambda^0((A,\varphi,\sigma)) = \left\{ x=(x_k) : lim_j \frac{1}{\lambda_j}\mu(K_\lambda^j((A,\varphi,\sigma),\varepsilon)) = 0, \text{uniformly in } i \right\}$$

If we take $A = I$ and $\varphi(x) = x$ respectively, then $S_\lambda^0((A,\varphi,\sigma))$ reduce to $S_{\sigma,\lambda}^0$ which was defined as follows:

$$S_{\sigma,\lambda}^0 = \left\{ x=(x_k) : lim_j \frac{1}{\lambda_j}|\{k \in I_j : |x_{\sigma^k(i)}| \geq \varepsilon\}| = 0, \text{uniformly in } i \right\}.$$

Finally we conclude this paper by stating the following theorem.

Theorem III.1. *If $\psi \prec \varphi$ then $S_\lambda^0((A,\psi)) \subset S_\lambda^0((A,\varphi))$.*

Proof. By assumption we have $\psi(|x_{\sigma^k(i)}|) \leq b\varphi(c|x_{\sigma^k(i)}|)$ and we have for all i,

$$\sum_{k=1}^{\infty} a_{nk}\psi(|x_{\sigma^k(i)}|) \leq b\sum_{k=1}^{\infty} a_{nk}\varphi(c|x_{\sigma^k(i)}|) \leq L\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|)$$

for $b, c > 0$, where the constant L is connected with properties of φ. Thus, the condition $\sum_{k=1}^{\infty} a_{nk}\psi(|x_{\sigma^k(i)}|) \geq \varepsilon$ implies the condition $\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{\sigma^k(i)}|) \geq \varepsilon$ and in consequence we get

$$\mu(K_\lambda^j((A,\varphi,\sigma),\varepsilon)) \subset \mu(K_\lambda^j((A,\psi,\sigma),\varepsilon))$$

and

$$lim_j \frac{1}{\lambda_j}\mu\Big(K_\lambda^j((A,\varphi,\sigma),\varepsilon)) \leq lim_j \frac{1}{\lambda_j}\mu(K_\lambda^j((A,\psi,\sigma),\varepsilon))\Big).$$

This completes the proof.

■

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